

Relations between classical phase-space distributions and Wigner functions for multiparticle production processes.

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Abstract

The effects of interpreting classical phase space distributions as Wigner functions, which is common in models of multiparticle production, are discussed. The temperature for the classical description is always higher than that for its Wigner function interpretation. A rough estimate shows that the corresponding correction is proportional to R^{-2} , where R is the radius of the interaction region, and that it is negligible for heavy ion scattering, but at the few percent level for e^+e^- annihilations.

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1 Introduction

Much work is being done on the femtoscopy of the interaction regions. One of the main problems is to find the space-time distribution of the set of the freeze-out points, i.e. of the points where the hadrons are finally freed. This is known to depend on the momenta of the particles, which significantly complicates the problem. For reviews of the work in this field see e.g. [1], [2], [3].

One can use several functions to describe the geometry of the interaction region in connection with the corresponding momentum distribution of the final state particles. The simplest is the classical phase space distribution for the particles at freeze-out $F(\mathbf{p}, \mathbf{x}, t)$. Many models provide just that. This is immediately seen when classical equations are being used as e.g. the Euler equations from hydrodynamics or the classical Boltzmann kinetic equation. For a discussion of a number of cascade models from this point of view see [4]. The classical approach is intuitive and most useful to get a general picture of the situation. In principle it contradicts quantum mechanics, because it is not possible to ascribe to a particle simultaneously a position in space and a momentum. In practice, however, often the quantum corrections are not very significant.

Another possibility is to use the Wigner function $W(\mathbf{p}, \mathbf{x}, t)$. This is well defined in quantum mechanics. Its relations to the density matrices in the momentum and coordinate representations are

$$W(\mathbf{K}, \mathbf{X}, t) = \int \frac{d^3q}{(2\pi)^3} \rho(\mathbf{K}, \mathbf{q}, t) e^{i\mathbf{q}\mathbf{X}}, \quad (1)$$

$$W(\mathbf{K}, \mathbf{X}, t) = \int \frac{d^3y}{(2\pi)^3} \tilde{\rho}(\mathbf{X}, \mathbf{y}, t) e^{-i\mathbf{K}\mathbf{y}}, \quad (2)$$

where

$$\mathbf{K} = \frac{1}{2}(\mathbf{p} + \mathbf{p}'), \quad \mathbf{q} = \mathbf{p} - \mathbf{p}', \quad \mathbf{X} = \frac{1}{2}(\mathbf{x} + \mathbf{x}'), \quad \mathbf{y} = \mathbf{x} - \mathbf{x}'. \quad (3)$$

In a rigorously understood sense [5],[6] the Wigner function is the best quantum-mechanical replacement for the classical phase-space density. Heisenberg's uncertainty principle is reflected by the inequality

$$|W(\mathbf{p}, \mathbf{x})| \leq \pi^{-3}, \quad (4)$$

which follows from the definition of the Wigner function. Wigner functions integrated over momenta give the correct space distributions and integrated over the space give the correct momentum distributions. The quantum mechanical averages of the type $\langle x^m p_x^n \rangle$ cannot, in general, be reliably calculated using the classical product with the Wigner function as weight, because they depend on the ordering of the noncommuting operators x and p_x . The averages calculated with the Wigner function always give the quantum mechanical average for the symmetrized (Weyl's ordering) product. E.g.

$$\int dx dp W(p, x) p_x^2 x = \frac{1}{4} \langle \hat{p}_x^2 \hat{x} + 2\hat{p}_x \hat{x} \hat{p}_x + \hat{x} \hat{p}_x^2 \rangle, \quad (5)$$

Here and in the following the hats are used to distinguish operators from the corresponding classical quantities. The most annoying feature of the Wigner function is that only in very exceptional cases it is nonnegative. In fact, for pure states the Wigner function is nowhere negative if and only if the corresponding wave function is a Gaussian [7]. Fortunately, for mixed states this implies that any average over Gaussians satisfying (4) can be a Wigner function, which is enough to reproduce almost any shape, provided there are no peaks violating the bound (4). According to the class of models described in the following section, in order to describe the multiparticle momentum distributions it is necessary to know the single particle density matrix $\rho_1(\mathbf{p}; \mathbf{p}')$. As seen from (2), this can be calculated when the Wigner function is known. It cannot, however, be obtained directly from the classical phase space distribution $F(\mathbf{p}, \mathbf{x}, t)$. Therefore, models which yield the classical density usually tacitly assume that it is sufficiently similar to the corresponding Wigner function to replace it in formula (2).

The purpose of the present paper is to study the relation between the functions $F(\mathbf{p}, \mathbf{x}, t)$ and $W(\mathbf{p}, \mathbf{x}, t)$. Our analysis suggests that this replacement is legitimate for heavy ion scattering, but overestimates the temperature of the system by several per cent for e^+e^- annihilations.

Still another possibility is to use the emission function [8], [9], [10], related to the density matrix by the relation

$$\rho(\mathbf{p}, \mathbf{p}') = N \int d^4 X S(K, X) e^{iqX}, \quad (6)$$

where q and X are four-vectors and N is a constant factor. This formula is applicable for times after freeze-out has been completed. Then, in the interaction representation, the density matrix does not depend on time any more. The emission function is particularly convenient when the time spread of the freeze-out process is of interest, In the present paper only simultaneous freeze-out will be considered, so the emission function will not be needed.

2 Simplifying assumptions

The multiparticle system just after freeze-out is in some complicated, highly correlated state. Therefore, in order to deal with it, it is necessary to introduce approximations. The simplest would be to neglect all the correlations. Then the diagonal elements of an n -particle density matrix in the momentum representation, which is what one needs to get the n -particle momentum distribution, would be given by the formula

$$\rho_{nu}(\mathbf{p}_1, \dots, \mathbf{p}_n; \mathbf{p}_1, \dots, \mathbf{p}_n) = \prod_{j=1}^n \rho_1(\mathbf{p}_j; \mathbf{p}_j), \quad (7)$$

where u in the subscript stands for uncorrelated. In this approximation, however, for n identical mesons there are no Bose-Einstein correlations. Since the Bose-Einstein correlations yield important information about the particle distributions in coordinate space, a better approximation must be used. The next choice [11] (for reviews see e.g. [1], [2], [3]) is to introduce proper symmetrization over the momenta of identical particles. Then for n identical mesons

$$\rho_n(\mathbf{p}_1, \dots, \mathbf{p}_n; \mathbf{p}_1, \dots, \mathbf{p}_n) = C_n \sum_P \prod_{j=1}^n \rho_1(\mathbf{p}_j; \mathbf{p}_{Pj}). \quad (8)$$

The summation is over all the permutations of the second arguments of ρ_1 . Symmetrizing also over the first arguments would just produce a constant factor $n!$, so there is no point in doing it. The normalization constant C_n is now necessary to ensure the proper normalization of ρ_n . With this choice, the single-particle and two-particle momentum distributions are

$$P(\mathbf{p}) = \overline{C}_1 \rho_1(\mathbf{p}; \mathbf{p}), \quad (9)$$

$$P(\mathbf{p}_1, \mathbf{p}_2) = \overline{C}_2 (P(\mathbf{p}_1)P(\mathbf{p}_2) + |\rho_1(\mathbf{p}_1; \mathbf{p}_2)|^2), \quad (10)$$

where \overline{C}_n are normalizing constants and the hermiticity of the density matrix ρ_1 has been used. Ansatz (8) leaves out the final state interactions. For a study of e.g. resonance production this would be unacceptable, but for analyses of the interaction regions the available methods of removing the effects of final state interactions from the data are good enough [3] and Ansatz (8) is widely used.

Let us assume further that, freeze-out for all the particles happens instantly and simultaneously at some time $t = 0$. With this assumption the emission function reduces to $\delta(t)W(\mathbf{p}, \mathbf{x})$ multiplied by a normalizing constant. Thus, there is no need to introduce an emission function besides the Wigner function, which greatly simplifies the discussion. Moreover, using the interaction representation one has, for $t \geq 0$, time independent density matrices and consequently time-independent Wigner functions. This assumption corresponds to a crude approximation. It would be better (cf. e.g. [3]) to assume that for each particle at its freeze-out its proper time τ has some fixed value, common for all the particles. Then, however, the problem of comparison would become much more difficult.

Finally, we assume that the particle density at freeze-out is given by the canonical distribution corresponding to some non-relativistic problem for noninteracting particles of mass m at temperature T in a force field corresponding to some potential $V(\mathbf{x})$. The most characteristic implication of this assumption is that the space extension of the interaction region increases with increasing temperature. This is the case for most models, but it is not a law of nature. For instance, stars get hotter when they shrink. The canonical distribution is being used in many models. The assumption of a non-relativistic potential is not realistic, but it is sufficiently general to reproduce any size and shape of the interaction region, so it seems sufficiently flexible to provide qualitatively reliable results.

3 The low- and high-temperature limits

In the low temperature limit, classically, the particle rests ($\mathbf{p} = \mathbf{0}$) at the minimum of the potential. Let us put there $\mathbf{x} = \mathbf{0}$. Thus,

$$F(\mathbf{p}, \mathbf{x}) = \delta(\mathbf{x})\delta(\mathbf{p}). \quad (11)$$

Because of the sharp peak this cannot be interpreted as a Wigner function. In order to get a candidate Wigner function, $F(\mathbf{p}, \mathbf{x})$ must be smeared.

The quantum-mechanical result is also easy to find. The particle must be in its ground state. Denoting the corresponding wave function $\psi_0(\mathbf{x})$ we get

$$W(\mathbf{p}, \mathbf{x}) = \int \frac{d^3y}{(2\pi)^3} \psi_0(\mathbf{x} + \frac{\mathbf{y}}{2}) \psi_0(\mathbf{x} - \frac{\mathbf{y}}{2}) e^{-i\mathbf{p}\mathbf{y}}. \quad (12)$$

Now both position and momentum are spread around the point $\mathbf{p} = \mathbf{0}, \mathbf{x} = \mathbf{0}$. In the theory of fluctuations this effect is referred to as quantum fluctuations. Formula (12) can be obtained by smearing (11). Therefore, smearing can be interpreted as a way of introducing quantum fluctuations. However, for each potential a different smearing prescription would be needed. Thus, at low temperatures the predictive power of the recipe: start with the classical distribution and smear it, is poor.

According to our assumptions, the classical distribution is in general

$$F(\mathbf{p}, \mathbf{x}) = N e^{-\frac{\beta \mathbf{p}^2}{2m} - \beta V(\mathbf{x})}. \quad (13)$$

Here and in the following N denotes a normalization factor. Where the normalization factors are of no interest, we will use the same notation for all of them. The quantum-mechanical density operator is

$$\hat{\rho} = N e^{-\frac{\beta \mathbf{p}^2}{2m} - \beta V(\hat{\mathbf{x}})}. \quad (14)$$

The difference between the classical and the quantum-mechanical expressions is that in the latter the kinetic energy does not commute with the potential energy. Let us note, however, that in the high-temperature limit β tends to zero. The commutator of the potential energy and kinetic energy terms in the exponent of (14) is proportional to β^2 and, therefore, is negligible. Accordingly, in the high-temperature limit the two description are equivalent, as will be demonstrated more rigorously latter.

The results from this section correspond to an effect which is well known from statistical physics. In the high-temperature limit the thermal fluctuations, common to the classical and quantum descriptions, usually dominate while in the low-temperature limit the quantum mechanical fluctuations, absent in the classical case, are the important ones.

4 The smearing density operators

When a classical phase-space distribution violates the bound (4), it is necessary to smear it. A way of doing this, is to introduce a (smearing) density operator as close as possible to the classical density distribution. Once a density operator is given, one can calculate from it a Wigner function which satisfies all the consistency conditions.

Let us try, as smearing density operator, the operator

$$\hat{\rho}_{sm} = Ne^{\frac{\beta\hat{\mathbf{p}}^2}{4m}} e^{-\beta V(\hat{\mathbf{x}})} e^{\frac{\beta\hat{\mathbf{p}}^2}{4m}}. \quad (15)$$

The kinetic energy term has been split in order to make this operator hermitian, as it should. The corresponding density matrix in the momentum representation is

$$\langle \mathbf{p} | \hat{\rho}_{int} | \mathbf{p}' \rangle = Ne^{-\frac{\beta}{2m}(\mathbf{K}^2 + \frac{1}{4}\mathbf{q}^2)} \int dx e^{-\beta V(\mathbf{x}) - i\mathbf{q}\mathbf{x}}, \quad (16)$$

and for the Wigner function one gets

$$W_{sm}(\mathbf{K}, \mathbf{X}) = Ne^{-\frac{\beta\mathbf{K}^2}{2m}} \int dx e^{-\beta V(\mathbf{x})} e^{-\frac{2m}{\beta}(\mathbf{X}-\mathbf{x})^2}. \quad (17)$$

In the high temperature limit, $\beta \rightarrow 0$, the second exponent in the integrand, taken with a suitable part of the normalizing factor, tends to $\delta(\mathbf{X} - \mathbf{x})$ and, after integrating over \mathbf{x} , one obtains the classical density distribution. For low temperatures, of course, no Wigner function can reproduce the classical distribution.

Let us consider as example the harmonic oscillator with $V(\mathbf{x}) = \frac{1}{2}k\mathbf{x}^2$. One gets

$$W_{sm}(\mathbf{p}, \mathbf{x}) = Ne^{-\frac{\beta\mathbf{p}^2}{2m^*} - \frac{\beta}{2}k^*\mathbf{x}^2}, \quad (18)$$

where

$$m^* = m, \quad k^* = \frac{1}{1 + \eta^2}k, \quad (19)$$

and

$$\eta = \frac{1}{2}\beta\omega. \quad (20)$$

The parameter $\omega = \sqrt{\frac{k}{m}}$ is the frequency of the oscillator. Note that with this smearing prescription the effective frequency $\omega_{eff} = \sqrt{\frac{k^*}{m^*}}$ depends on temperature. In order to get after smearing a distribution identical with $F(\mathbf{p}, \mathbf{x})$ one would have to make before smearing the substitution

$$k \rightarrow (1 + \eta^2)k. \quad (21)$$

At high temperatures $k^* \approx k$ and the classical result is reproduced. At low temperatures, when η is large, $k^* \ll k$ and the x -distribution is smeared which avoids the contradiction to Heisenberg's uncertainty principle.

As another example let us choose

$$\hat{\rho}_{sm} = Ne^{-\frac{1}{2}\beta V(\hat{\mathbf{x}})}e^{-\frac{\beta \hat{\mathbf{p}}^2}{2m}}e^{-\frac{1}{2}\beta V(\hat{\mathbf{x}})}. \quad (22)$$

For the harmonic oscillator this yields again formula (18), but with

$$m^* = (1 + \eta^2) m, \quad k^* = k. \quad (23)$$

This time the smearing is in momentum space. A popular smearing prescription [12], [13] is

$$\hat{\rho}_{sm} = \int dp dx F(\mathbf{p}, \mathbf{x}, t) |\psi(\mathbf{p}, \mathbf{x})\rangle \langle \psi(\mathbf{p}, \mathbf{x})|, \quad (24)$$

where the state vector $|\psi(p, x)\rangle$ represents a bound state of one particle with positions close to \mathbf{x} and momenta close to \mathbf{p} .

It is seen that various choices of $\hat{\rho}_{sm}$ correspond to various smearing prescriptions. Each of them gives a reasonable Wigner function, but only with (14) chosen as the smearing density operator the correct Wigner function is obtained. We will compare now in the general case the smearing density operator (15) with the exact one (14).

5 Effective Hamiltonian

The results obtained in the preceding section for the harmonic oscillator can be generalized to other potentials. One always finds that the smeared Wigner function corresponds to some Hamiltonian, but in general not to the true one for the system being studied. We will call this Hamiltonian effective Hamiltonian. It is defined by the relation

$$\hat{\rho}_{sm} = Ne^{-\beta\hat{H}_{eff}}. \quad (25)$$

The smearing density operator (15) has the form

$$\hat{\rho}_{sm} = Ne^{\hat{X}}e^{\hat{Y}}e^{\hat{X}}, \quad (26)$$

Thus, ignoring the irrelevant constants $\log N$,

$$-\beta\hat{H}_{eff} = \log\left(e^{\hat{X}}e^{\hat{Y}}e^{\hat{X}}\right). \quad (27)$$

The right-hand side can be evaluated from a simple extension of the famous Baker-Campbell-Hausdorff formula. The result is a series, in general infinite, of iterated commutators constructed from the operators \hat{X} and \hat{Y} . Since both \hat{X} and \hat{Y} are proportional to β , this is a power series expansion in β . An elegant and convenient method for calculating the coefficients of this series for a more general case, i.e. for $\log\left(e^{\hat{X}}e^{\hat{Y}}e^{\hat{Z}}\right)$, has been described in ref. [15]. In our case an additional simplification occurs. Note that the operator $e^{-\hat{X}}e^{-\hat{Y}}e^{-\hat{X}}$ is the inverse of the operator $e^{\hat{X}}e^{\hat{Y}}e^{\hat{X}}$. Therefore, its logarithm equals $+\beta\hat{H}_{eff}$. On the other hand, the expansion of this logarithm can be obtained by taking the expansion for (27) and changing the signs of all the \hat{X} -s and \hat{Y} -s. These two prescriptions are consistent if and only if all the commutators with even numbers of factors have coefficients zero. For instance, for the smearing density operator (15) one finds

$$\hat{H}_{eff} = \hat{H} - \frac{\beta^2}{6} \left[\hat{H}, \left[\frac{\hat{\mathbf{p}}^2}{4m}, V(\hat{\mathbf{x}}) \right] \right] + \dots, \quad (28)$$

where $\hat{H} = \frac{\hat{\mathbf{p}}^2}{2m} + V(\hat{\mathbf{x}})$ is the original Hamiltonian. The contribution of the single commutator vanishes as it should.

In particular, for the harmonic oscillator

$$\hat{H}_{eff} = \hat{H} + \frac{\eta^2}{3} \left(\frac{\hat{\mathbf{p}}^2}{2m} - k\hat{\mathbf{x}}^2 \right) + \dots. \quad (29)$$

Using a program in *MATHEMATICA* given in [15] it is easy to calculate more terms of this series. In order to get an effective Hamiltonian corresponding to the original phase-space density, and not to its smeared version, one must make in the Hamiltonian on the right-hand side the substitution (21). This yields the Hamiltonian

$$\hat{H}^* = \left(1 + \frac{1}{3}\eta^2\right) \hat{H}, \quad (30)$$

which reproduces, to second order in η the classical distribution (13).

For the harmonic oscillator, it is easy to compare directly, without using a smearing density operator, the Wigner function with the corresponding classical distribution. This is discussed in the following section.

6 Classical density and Wigner function for the harmonic oscillator

For a harmonic oscillator at temperature T , the Wigner function, or equivalently the density matrix, has been calculated by a variety of methods [16], [6], [17]. The result is

$$W(\mathbf{p}, \mathbf{x}) = N e^{-\beta \frac{\tanh \eta}{\eta} (\frac{\mathbf{p}^2}{2m} + \frac{1}{2}k\mathbf{x}^2)}. \quad (31)$$

This is to be compared with the corresponding classical density

$$F(\mathbf{p}, \mathbf{x}) = \left(\frac{\eta_{class}}{\pi}\right)^3 e^{-\beta_{class}(\frac{\mathbf{p}^2}{2m} + \frac{1}{2}k\mathbf{x}^2)}. \quad (32)$$

According to condition (4), if $\eta_{class} \equiv \frac{1}{2}\beta_{class}\omega > 1$ the classical density must be smeared before being interpreted as a Wigner function. The distributions (31) and (32) coincide, if

$$\eta_{class} = \tanh \eta. \quad (33)$$

At high temperatures, where η and η_{class} are both small, $\eta_{class} \approx \eta$ and there is no harm in interpreting the classical distribution as a Wigner function. At low temperatures, however, η can be arbitrarily large, while η_{class} never exceeds unity. Then, interpreting the classical distribution as a Wigner function can lead to serious errors.

An obvious question is, where on this scale are situated the temperatures in the range of some (100 – 200) MeV relevant for high energy scattering? The difficulty is that, what matters is the temperature in units of ω , and ω is not known. In order to get a rough estimate, let us make the admittedly crude assumption that the results for the harmonic oscillator can be used as a guide. For the harmonic oscillator

$$\sigma^2(p_i) = \frac{\sqrt{km}}{2 \tanh \eta}, \quad \sigma^2(x_i) = \frac{1}{2 \sqrt{km} \tanh \eta}, \quad i = 1, 2, 3. \quad (34)$$

This yields

$$\tanh \eta = \frac{1}{2\sqrt{\sigma^2(x_i)}\sqrt{\sigma^2(p_i)}}. \quad (35)$$

Choosing a value typical for high energy scattering, $\sqrt{\sigma^2(p_i)} = 300\text{MeV}$, one gets as an approximation, which is very good when $|\eta - \eta_{class}|$ is small,

$$T = \left(1 - \frac{0.036}{\sigma^2(x_i)}\right) T_{class}, \quad (36)$$

where $\sigma^2(x_i)$ should be expressed in squared fermis. It is seen that for heavy ion high energy scattering, where typically $\sqrt{\sigma^2(x_i)} \approx 5\text{fm}$, the correction is negligible. For e^+e^- annihilations, however, where $\sqrt{\sigma^2(x_i)}$ can be as small as 0.7fm , the correction is about seven percent. We conclude that, interpreting the classical distribution as a Wigner function one always finds that the classical temperature is higher than the one corresponding to the Wigner function interpretation. Qualitatively this conclusion seems unavoidable. The quantum fluctuations are not included in the classical description. In order to reproduce their effect it is necessary to increase the thermal fluctuations, which means increasing the temperature. The corresponding correction is probably negligible for heavy ion collisions, but may be at the few percent level for e^+e^- annihilations.

In order to obtain a Wigner function of the form (13) with $\beta_{class} = \beta$, one has to start with the Hamiltonian

$$\hat{H}^* = \frac{\eta}{\tanh \eta} \hat{H} = \left(1 + \frac{1}{3}\eta^2 + \dots\right) \hat{H}, \quad (37)$$

which agrees with (30) to second order in η .

7 Conclusions

Numerous models provide classical phase space distributions for the particles produced in multiparticle production processes. When describing Bose-Einstein correlations these densities, sometimes smeared, are being used as

if they were Wigner functions. Therefore, it is an interesting question how close, in situations encountered in particle physics, are the classical phase space distributions to their corresponding Wigner functions.

Converting a classical phase space distribution to a Wigner function, when temperature is not very high, one should in principle consider quantum fluctuations. The simplest way is to assume that they are negligible. Our discussion, based on the analogy with the harmonic oscillator, suggests that this could be legitimate for high-energy heavy ion collisions, but probably not for e^+e^- annihilations.

In general, quantum fluctuations are negligible at high temperatures and important at low temperatures. For a given potential this means that, they are important when the interaction region is small, and unimportant when it is large. For the specific model discussed in the present paper, the correction goes like R^{-2} as seen from formula (36). The correction always reduces the inferred temperature of the system.

For potentials more complicated than that of the harmonic oscillator, it is convenient to perform the comparison of the classical phase-space distribution with the corresponding Wigner function in two steps. First one introduces a smearing density operator, which should provide a Wigner function easy to compare with the classical distribution. This is equivalent to the familiar smearing and yields a Wigner function which satisfies all the general consistency conditions. It can be done in an infinity of ways. Three are described in the text. The introduction of the smearing density operator is equivalent to the introduction of an effective hamiltonian which yields the same Wigner function as the smeared density operator. In the first two examples discussed here, using the Baker-Campell-Hausdorff formula, one can obtain this effective Hamiltonian as a power series in β . The leading term is the true Hamiltonian which confirms that in the high temperature limit ($\beta \rightarrow 0$) the crude estimate of the quantum fluctuations, as done by introducing smearing, is good enough. This is implied by the fact that quantum fluctuations are negligible.

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